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On the influence of the electric permeability on an interface crack in a piezoelectric bimaterial compound

V.B. Govorukha [a,*], V.V. Loboda [b], M. Kamlah [c]

[a] *Department of Computational Mathematics, Dniepropetrovsk National University, Nauchny str. 13, Dniepropetrovsk 49050, Ukraine*

[b] *Department of Theoretical and Applied Mechanics, Dniepropetrovsk National University, Nauchny str. 13, Dniepropetrovsk 49050, Ukraine*

[c] *Forschungszentrum Karlsruhe, Institut for Materials Research II, Postfach 3640, Karlsruhe 76021, Germany*

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Abstract

An interface crack between two semi-infinite piezoelectric spaces under the action of remote mixed mode loading and electric flux is considered. The properties of the materials, loading and crack geometry admit to consider a two-dimensional problem in the plane perpendicular to the crack front. The crack is assumed to be free from mechanical loading and the limited permeable electric condition holds true. Assuming the electric flux is constant along the crack area, using the known presentations of all electromechanical fields via a piecewise holomorphic vector function, the problem is reduced to a vector Hilbert problem and solved in an analytical way. Clear analytical expressions for stresses and electric displacement as well as for stress and electric intensity factors are derived. As a particular case, a crack in a homogeneous piezoelectric material is considered and exact analytical formulae are presented for this case. The numerical analysis of the obtained formulae showed that for small values of the electric flux the model of a completely permeable crack can be used for any real crack permeability's. The validity of such an approximation decreases with increase in the mechanical loading and especially of the electric flux.

Keywords: Limited permeable interface crack; Piezoelectric bimaterial compound

* Corresponding author. Tel.: +380 562 451411; fax: +380 56 7765833.
E-mail addresses: govor@amsu.dnp.ukrpack.net, govorukhavb@yahoo.com (V.B. Govorukha).

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1. Introduction

Because of an intrinsic coupled electromechanical behavior, piezoelectric materials are intensively used in engineering as sensors, transducers and actuators. But piezoelectric materials often contain many micro defects such as cracks which reduce their strength. The problem of a crack in a piezoelectric material has been actively studied because of its importance with special attention to the choosing of correct electric conditions at the crack faces. Because of the complexity of the problem, the extreme cases of electrically permeable (Parton, 1976; Parton and Kudryavtsev, 1988) and electrically impermeable cracks (Suo et al., 1992) were mostly used for the investigation of interface cracks in piezoelectric bimaterial compounds. Both types of these electric conditions were studied in papers by Govorukha and Loboda (2000) and Govorukha et al. (2000).

The validity of a simplified electric boundary condition at the crack faces in a homogeneous piezoelectric material has been investigated by Dunn (1994), Sosa and Khutoryansky (1996), Kogan et al. (1996), Zhang et al. (1998), Gao and Fan (1999) by considering a slit crack as a limiting case of an elliptical hole or an inclusion. Taking into account the exact electric field in the mentioned hole or inclusion, they arrived at the conclusion that the assumption of a permeable crack is more realistic than that of an impermeable crack and moreover, that the latter assumption leads to the appearance of an additional singularity of the electric displacement at the crack tip which depends only on the electric loading.

Another way of considering the electric permeability of the crack medium was suggested by Hao and Shen (1994). They used the so-called limited permeable boundary condition in which the electric permeability of the environment in the crack gap was taken into account by considering the crack as a condensator. The same way of electric permeability modeling was used in paper by Gruebner et al. (2003), in which the finite element method was used. The same method together with analytical approach was applied to the analysis of the similar problem by Wang and May (2003). All these papers were devoted to the investigations of cracks in a homogeneous material.

Investigation similar to (Hao and Shen, 1994; Gruebner et al., 2003) for a crack in the interface of a piezoelectric bimaterial compound are not known to the authors of this paper. In fact, two simplified cases of the boundary conditions at the interface crack faces are actively used now, i.e., electrically impermeable crack and electrically permeable crack, respectively (Suo et al., 1992; Herrmann and Loboda, 2000; Herrmann et al., 2001; Gao and Wang, 2000).

In the present study, the limited permeable assumption (Hao and Shen, 1994) is applied to the analysis of an interface crack in a piezoelectric bimaterial compound. An analytical approach for a compound consisting of two linear piezoelectric materials has been used. As a special case, a limited permeable crack in a homogeneous material has been studied as well.

2. General solution of the basic equations

The constitutive relations for a linear piezoelectric material in the absence of body forces and free charges can be presented in the form by Pak (1992):

$$\Pi_{iJ} = E_{iJKl} V_{K,l}, \tag{1}$$

$$\Pi_{iJ,i} = 0, \tag{2}$$

where

$$V_K = \begin{cases} u_k, & K = 1,2,3, \\ \varphi, & K = 4, \end{cases} \tag{3}$$

$$\Pi_{iJ} = \begin{cases} \sigma_{ij}, & i,J = 1,2,3, \\ D_i, & i = 1,2,3;\ J = 4 \end{cases} \tag{4}$$

and

$$E_{iJKl} = \begin{cases} c_{ijkl}, & J,K = 1,2,3, \\ e_{lij}, & J = 1,2,3;\ K = 4, \\ e_{ikl}, & K = 1,2,3;\ J = 4, \\ -\varepsilon_{il}, & J = K = 4 \end{cases} \tag{5}$$

and u_k, φ, σ_{ij} and D_i are the elastic displacements, electric potential, stresses and electric displacements, respectively. Furthermore, c_{ijkl}, e_{lij} and ε_{ij} are the elastic, piezoelectric and dielectric constants, respectively. Small subscripts in (1)–(5) and afterwards are always ranging from 1 to 3, capital subscripts are ranging from 1 to 4 and Einstein's summation convention is used in (1) and (2).

In papers by Herrmann and Loboda (2000) and Herrmann et al. (2001) similar to the solution by Suo et al. (1992) the following representations have been derived for a piezoelectric bimaterial plane

$$[\mathbf{V}'(x_1,0)] = \mathbf{W}^+(x_1) - \mathbf{W}^-(x_1), \tag{6}$$

$$\mathbf{t}^{(1)}(x_1,0) = \mathbf{G}\mathbf{W}^+(x_1) - \overline{\mathbf{G}}\mathbf{W}^-(x_1), \tag{7}$$

where

$$[\mathbf{V}'(x_1,0)] = \mathbf{V}'^{(1)}(x_1,0) - \mathbf{V}'^{(2)}(x_1,0), \tag{8}$$

and $\mathbf{G} = \mathbf{B}^{(1)}\mathbf{D}^{-1}$, $\mathbf{D} = \mathbf{A}^{(1)} - \overline{\mathbf{A}}^{(2)}(\overline{\mathbf{B}}^{(2)})^{-1}\mathbf{B}^{(1)}$, $\mathbf{W}^+(x_1) = \mathbf{W}(x_1 + \mathrm{i}0)$, $\mathbf{W}^-(x_1) = \mathbf{W}(x_1 - \mathrm{i}0)$; $\mathbf{A}^{(m)}$, $\mathbf{B}^{(m)}$ are known matrices (Suo et al., 1992) related to the upper ($m = 1$) and lower ($m = 2$) half-planes, respectively; $\mathbf{V} = [u_1, u_2, u_3, \varphi]^{\mathrm{T}}$ and $\mathbf{t} = [\sigma_{13}, \sigma_{23}, \sigma_{33}, D_3]^{\mathrm{T}}$. It is worth to note that the unknown vector-function $\mathbf{W}(z) = [W_1(z), W_2(z), W_3(z), W_4(z)]^{\mathrm{T}}$ is analytic in the whole plane including the bonded parts of the material interface ($z = x_1 + \mathrm{i}x_3$, $\mathrm{i} = \sqrt{-1}$). Moreover the $[4 \times 4]$ bimaterial matrix $\mathbf{G}$ and the vector function $\mathbf{W}(z)$ are related to the matrix $\mathbf{H}$ and the vector function $\mathbf{h}(z)$ in Suo et al. (1992) as $\mathrm{i}\mathbf{G}^{-1} = \mathbf{H}$, $\mathbf{W}(z) = -\mathrm{i}\mathbf{H}\mathbf{h}(z)$, respectively.

These representations (6) and (7) are useful for the formulation of linear piezoelectric problems concerning the different conditions at the interface of a semi-infinite plane. Particularly, Herrmann and Loboda (2000) and Herrmann et al. (2001) applied these representations for the investigations of an interface crack with a contact zone providing electrically permeable and electrically impermeable conditions at its faces, respectively.

3. Formulation of the problem

Consider an interface crack $-b \leqslant x_1 \leqslant b$, $x_3 = 0$ between two semi-infinite piezoceramic spaces $x_3 > 0$ (with a matrix $E^{(1)}_{iJKl}$ of the physical properties) and $x_3 < 0$ (with a matrix $E^{(2)}_{iJKl}$) having both the symmetry class of *6mm* with the poling direction x_3. The loading at infinity is given by $\sigma^{(m)}_{33} = \sigma$, $\sigma^{(m)}_{13} = \tau$, $\sigma^{(m)}_{11} = \sigma^{\infty}_{xxm}$, $D^{(m)}_3 = d$, $D^{(m)}_1 = D^{\infty}_{xm}$ ($m = 1$ stands for the upper domain, and $m = 2$ for the lower one). It produces stresses and displacements which satisfy continuity conditions at the interface. Because, the load does not depend on the coordinate x_2, the plane strain problem in the (x_1, x_3) plane depicted in Fig. 1 can be considered. Neglecting the small zones of oscillation (Parton, 1976) we will assume that the crack is completely open and its faces are free of prescribed mechanical loading and electric charges. Moreover, we assume that the electric field inside the crack can be found as $E_a = -(\varphi^+ - \varphi^-)/(u_3^+ - u_3^-)$. Taking into account that $D_3 = \varepsilon_a \mathrm{E}_a$, one arrives to the electric condition $D_3 = -\varepsilon_a(\varphi^+ - \varphi^-)/(u_3^+ - u_3^-)$ along the crack region which was analyzed earlier by Hao and Shen (1994). Thus the boundary conditions at the material interface can be written as

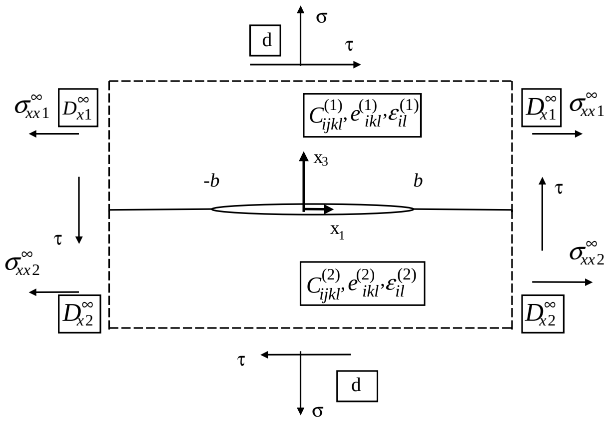

Fig. 1. Piezoelectric bimaterial plane with a limited permeable interface crack.

$$\text{for } x_1 \notin (-b,b): \quad [\mathbf{V}(x_1,0)] = 0, \quad [\mathbf{t}(x_1,0)] = 0, \tag{9}$$

$$\begin{aligned} \text{for } x_1 \in (-b,b): \quad & \sigma_{13}^{(m)}(x_1,0) = 0, \quad \sigma_{33}^{(m)}(x_1,0) = 0, \\ & [D_3(x_1,0)] = 0, \quad D_3[u_3(x_1,0)] = -\varepsilon_a[\varphi(x_1,0)], \end{aligned} \tag{10}$$

where ε_a is the permeability of the crack medium and the square brackets mean the jump of the corresponding function across the material interface.

On the base of Eqs. (6) and (7) the relation

$$\sigma_{33}^{(1)}(x_1,0) + m_{j4}D_3^{(1)}(x_1,0) + \mathrm{i}m_{j1}\sigma_{13}^{(1)}(x_1,0) = F_j^+(x_1) + \gamma_j F_j^-(x_1) \tag{11}$$

has been obtained by Herrmann et al. (2001), where

$$F_j(z) = n_{j1}W_1(z) + \mathrm{i}[n_{j3}W_3(z) + n_{j4}W_4(z)] \tag{12}$$

and m_{jl}, n_{jl}, γ_j $(j,l = 1,3,4)$ depend on the material constants and have real values for certain classes of piezoceramics. Besides, the functions $F_j(z)$ are analytic in the whole plane including the bonded parts of the material interface.

4. Determination of the electric flux over the crack region

We now assume that the electric flux is constant along the crack faces, i.e.

$$D_3^+(x_1,0) = D_3^-(x_1,0) = D \quad \text{for } x_1 \in (-b,b). \tag{13}$$

The validity of this assumption will be approximately confirmed later.

Eqs. (11) and (13) together with the interface conditions (10) lead to

$$F_j^+(x_1) + \gamma_j F_j^-(x_1) = m_{j4}D \quad (j = 1,3,4) \text{ for } x_1 \in (-b,b), \tag{14}$$

which is a Riemann problem in the sense by Muskhelisvili (1953). For $x_1 \notin (-b,b)$, the relation $F_j^+(x_1) = F_j^-(x_1)$ is valid, and therefore one can write by means of Eq. (11) and the remote prescribed electromechanical loads at infinity the conditions

$$F_j(z)|_{z\to\infty} = \tilde{\sigma}_j - \mathrm{i}\tilde{\tau}_j \tag{15}$$

for the functions $F_j(z)$, where $\tilde{\sigma}_j = \frac{1}{r_j}(\sigma + m_{j4}d)$, $\tilde{\tau}_j = -m_{j1}\tau/r_j$, $r_j = (1+\gamma_j)$ $(j = 1,3,4)$.

By introducing the new function

$$\Phi_j(z) = F_j(z) - \frac{m_{j4}D}{1+\gamma_j} \tag{16}$$

having the same properties as $F_j(z)$, Eqs. (14) and (15) take the form

$$\Phi_j^+(x_1) + \gamma_j \Phi_j^-(x_1) = 0 \quad (j = 1, 3, 4) \text{ for } x_1 \in (-b, b), \tag{17}$$

$$\Phi_j(z)|_{z\to\infty} = \sigma_j^* - \mathrm{i}\tau_j^*, \tag{18}$$

where $\sigma_j^* = \frac{1}{r_j}[\sigma + m_{j4}(d - D)], \quad \tau_j^* = \tilde{\tau}_j, (j = 1, 3, 4)$.

According to the results by Muskhelisvili (1953) the solution of the problem (17) and (18) has the form

$$\Phi_j(z) = X_j(z)(\sigma_j^* - \mathrm{i}\tau_j^*)(z - 2\mathrm{i}b\varepsilon_j), \tag{19}$$

where $X_j(z) = (z+b)^{-1/2+\mathrm{i}\varepsilon_j}(z-b)^{-1/2-\mathrm{i}\varepsilon_j}$, $\varepsilon_j = \frac{\ln\gamma_j}{2\pi}$.

By use of Eq. (12) the relation

$$n_{j1}[u_1'(x_1, 0)] + \mathrm{i}\{n_{j3}[u_3'(x_1, 0)] + n_{j4}[\varphi'(x_1, 0)]\} = F_j^+(x_1) - F_j^-(x_1) \tag{20}$$

can be derived. Moreover, since from (17) $\Phi_j^-(x_1) = -\Phi_j^+(x_1)/\gamma_j$ for $x_1 \in (-b, b)$ and therefore $F_j^+(x_1) - F_j^-(x_1) = \frac{\gamma_j+1}{\gamma_j}\Phi_j^+(x_1)$ holds, one can get

$$n_{j1}[u_1'(x_1, 0)] + \mathrm{i}\{n_{j3}[u_3'(x_1, 0)] + n_{j4}[\varphi'(x_1, 0)]\} = \frac{\gamma_j+1}{\gamma_j}(\sigma_j^* - \mathrm{i}\tau_j^*)(x_1+b)^{-1/2+\mathrm{i}\varepsilon_j}(x_1-b)^{-1/2-\mathrm{i}\varepsilon_j}(x_1 - 2\mathrm{i}b\varepsilon_j). \tag{21}$$

By integrating this equation, one arrives to the formula

$$n_{j1}[u_1(x_1, 0)] + \mathrm{i}\{n_{j3}[u_3(x_1, 0)] + n_{j4}[\varphi(x_1, 0)]\} = \frac{\gamma_j+1}{\gamma_j}(\sigma_j^* - \mathrm{i}\tau_j^*)\left(\frac{x_1+b}{x_1-b}\right)^{\mathrm{i}\varepsilon_j}\sqrt{x_1^2 - b^2} \quad \text{for } x_1 \in (-b, b). \tag{22}$$

The analysis shows that for the ceramics of the symmetry class of *6mm* with the poling direction x_3, the relations $n_{41} = 0$, $\varepsilon_4 = 0$, $\gamma_4 = 1$ are valid (Herrmann et al., 2001). Because of this, the equations

$$n_{13}[u_3(x_1, 0)] + n_{14}[\varphi(x_1, 0)] = \mathrm{Im}\left\{\frac{\gamma_1+1}{\gamma_1}(\sigma_1^* - \mathrm{i}\tau_1^*)\left(\frac{x_1+b}{x_1-b}\right)^{\mathrm{i}\varepsilon_1}\sqrt{x_1^2 - b^2}\right\}, \tag{23}$$

$$n_{43}[u_3(x_1, 0)] + n_{44}[\varphi(x_1, 0)] = -2\mathrm{i}(\sigma_4^* - \mathrm{i}\tau_4^*)\sqrt{x_1^2 - b^2} \tag{24}$$

can be derived from (22) for $x_1 \in (-b, b)$.

These relations are a system of linear algebraic equations for $[u_3\ (x_1, 0)]$ and $[\varphi(x_1, 0)]$ leading to the solution

$$\begin{aligned} [u_3(x_1, 0)] &= \Delta^{-1}\{n_{44}H_1(x_1) - n_{14}H_2(x_1)\}, \\ [\varphi(x_1, 0)] &= \Delta^{-1}\{-n_{43}H_1(x_1) + n_{13}H_2(x_1)\}, \end{aligned} \tag{25}$$

where

$$H_1(x_1) = \frac{\gamma_1+1}{\gamma_1}(\sigma_1^*\cos\alpha + \tau_1^*\sin\alpha)\sqrt{b^2 - x_1^2}, \quad H_2(x_1) = 2\sigma_4^*\sqrt{b^2 - x_1^2},$$

$$\alpha = \varepsilon_1 \ln\left(\frac{b+x_1}{b-x_1}\right), \quad \Delta = n_{13}n_{44} - n_{43}n_{14}.$$

Substituting Eq. (25) into the second of relations (11), we arrive to the equation

$$D = \varepsilon_a \frac{\gamma_0(\sigma_1^* \cos\alpha + \tau_1^* \sin\alpha)n_{43} - 2\sigma_4^* n_{13}}{\gamma_0(\sigma_1^* \cos\alpha + \tau_1^* \sin\alpha)n_{44} - 2\sigma_4^* n_{14}} \quad \text{for } x_1 \in (-b, b) \tag{26}$$

or the electric flux D at the crack faces. Taking into account that σ_1^* and τ_1^* linearly depend on D, relation (26) is quadratic equation for D.

It should be mentioned that cos(α) and sin(α) in Eq. (26) depend on x_1 and therefore D is not a constant along the interval (−1, 1), in general. However for compounds of existing piezoelectric materials, the value of ε_1 is rather small (Parton and Kudryavtsev, 1988) and therefore the functions cos(α) and sin(α) are almost constant in the interval (−1, 1). For example for the bimaterial compound PZT4/PZT5 the value $\varepsilon_1 = 0.01290$ is found. Thus, the variations of cos(α) and sin(α) for this bimaterial compound as well as for others are negligible and the approximations cos(α) ≈ 1, sin(α) ≈ 0 can be assumed with high accuracy. In this case, Eq. (26) takes the form

$$D = \varepsilon_a \frac{\gamma_0 \sigma_1^* n_{43} - 2\sigma_4^* n_{13}}{\gamma_0 \sigma_1^* n_{44} - 2\sigma_4^* n_{14}} \tag{27}$$

and can be reformulated as

$$\eta_1 D^2 + \eta_2 D + \eta_3 = 0, \tag{28}$$

where

$$\begin{aligned} &\eta_1 = 2r_1 m_{44} n_{14} - \gamma_0 r_4 m_{14} n_{44}, \quad \eta_2 = \gamma_0 r_4 (s_1 n_{44} + \varepsilon_a m_{14} n_{43}) - 2r_1 (s_4 n_{14} + \varepsilon_a m_{44} n_{13}), \\ &\eta_3 = 2r_1 \varepsilon_a n_{13} s_4 - \gamma_0 r_4 \varepsilon_a n_{43} s_1, \quad s_1 = \sigma + m_{14} d, \quad s_4 = \sigma + m_{44} d. \end{aligned} \tag{29}$$

An analytical investigation and the numerical evaluation of Eq. (28) showed that one root of this equation is not in agreement with physical consideration. For example, it remains finite and relatively large for $\varepsilon_a \to 0$, i.e. in case of an impermeable crack. Therefore, it is easy to choose the physically relevant root of (28) and only this root is used in the following analysis. Thus, the solution of Eq. (28) yields the electric flux D for $x_1 \in (-b, b)$, which is constant according to our assumption.

5. Electromechanical fields and intensity factors

By means of relations (11), (16) and (19) and in view of the properties of the matrix **m** the stresses and the electric displacement for $x_1 > b$ can be written in the form

$$\begin{aligned} &\sigma_{33}^{(1)}(x_1, 0) + m_{14} D_3^{(1)}(x_1, 0) + \mathrm{i} m_{11} \sigma_{13}^{(1)}(x_1, 0) \\ &\quad = (1 + \gamma_1)(\sigma_1^* - \mathrm{i}\tau_1^*)(x_1 - 2\mathrm{i}b\varepsilon_1)(x_1 + b)^{-1/2 + \mathrm{i}\varepsilon_1}(x_1 - b)^{-1/2 - \mathrm{i}\varepsilon_1} + m_{14} D, \end{aligned} \tag{30}$$

$$\sigma_{33}^{(1)}(x_1, 0) + m_{44} D_3^{(1)}(x_1, 0) = \frac{2\sigma_4^* x_1}{\sqrt{x_1^2 - b^2}} + m_{14} D. \tag{31}$$

From the system (30) and (31), expressions for $\sigma_{33}^{(1)}(x_1, 0)$, $D_3^{(1)}(x_1, 0)$ and $\sigma_{13}^{(1)}(x_1, 0)$ for $x_1 > b$ can be computed.

The intensity factors (IFs) at the point b are defined as (Herrmann et al., 2001)

$$\begin{aligned} &K_1 + m_{14} K_4 - \mathrm{i} m_{11} K_2 \\ &\quad = \lim_{x_1 \to b+0} \sqrt{2\pi(x_1 - b)}(x_1 - b)^{\mathrm{i}\varepsilon_1} [\sigma_{33}^{(1)}(x_1, 0) + m_{14} D_3^{(1)}(x_1, 0) + \mathrm{i} m_{11} \sigma_{13}^{(1)}(x_1, 0)], \end{aligned} \tag{32}$$

$$K_1 + m_{44} K_4 = \lim_{x_1 \to b+0} \sqrt{2\pi(x_1 - b)} [\sigma_{33}^{(1)}(x_1, 0) + m_{44} D_3^{(1)}(x_1, 0)]. \tag{33}$$

Using (30), (31) one gets for $x_1 \to b + 0$

$$K_1 + m_{14}K_4 - \mathrm{i}m_{11}K_2 = \sqrt{\frac{l\pi}{2}}(1 - 2\mathrm{i}\varepsilon_1)[\sigma + m_{14}(d - D) - \mathrm{i}m_{11}\tau]\mathrm{e}^{\mathrm{i}\psi}, \tag{34}$$

$$K_1 + m_{44}K_4 = \sqrt{\frac{l\pi}{2}}[\sigma + m_{44}(d - D)], \tag{35}$$

where $\psi = \varepsilon \ln l$, $\alpha = \frac{(\gamma_1+1)^2}{4\gamma_1}$ and $l = 2b$ is the crack length. From the formulae (34), (35) the analytical expressions for K_1, K_2 and K_4 can be derived.

For comparison, consider now the extreme cases of electrically permeable and electrically impermeable cracks. An electrically impermeable crack has been analyzed by Herrmann et al. (2001) and the corresponding results directly follow from the formulae (30)–(35) for $\varepsilon_a \to 0$ ($D \to 0$). For the case of an electrically permeable crack combining the Eqs. (38) and (71) of the paper by Herrmann and Loboda (2000) one gets for $x_1 \to b - 0$

$$D_3^{(\mathrm{perm})}(x_1, 0) = -\left(g_{41} - \frac{g_{43}g_{31}}{g_{33}}\right)\frac{2\sqrt{\alpha_p}b}{r_p\sqrt{b - x_1}}[(2\varepsilon_p\sigma - m_p\tau)\cos\omega_p + (\sigma + 2\varepsilon_p m_p\tau)\sin\omega_p] + d - \frac{g_{43}}{g_{33}}\sigma, \tag{36}$$

where $\omega_p = \varepsilon_p \ln\frac{b-x_1}{l}$ and, furthermore, g_{ij} are the components of the matrix $\mathbf{G}$ and ε_p, m_p, α_p, r_p the constants for a permeable crack defined by Herrmann and Loboda (2000). In particular, for a homogeneous piezoelectric material one finds $D_3^{(\mathrm{perm})}(x_1, 0) = d - g_{43}g_{33}^{-1}\sigma$, which completely coincides with the corresponding expression of the paper by Gao and Wang (2000).

In Fig. 2, the variation of $D_3^{(\mathrm{perm})}(x_1, 0)$ along the crack region for the electrically permeable crack assumption defined by formula (36) (line I) and the value of D from the solution of (28) for $\varepsilon_a = 4000\varepsilon_0$ (line II) are shown ($\varepsilon_0 = 8.85 \times 10^{-12}$ C/V m). The bimaterial compound PZT4/PZT5 was used and $\sigma = 10$ MPa, $\tau = 0$ MPa, $d = 0.01$ C/m^2. The obtained results show only in the immediate neighborhood of the crack tip a visible difference between $D_3^{(\mathrm{perm})}(x_1, 0)$ and D. For the remaining part of $(-b, b)$ their values coincide. This supports our assumption $\cos(\alpha) \approx 1$, $\sin(\alpha) \approx 0$ for the calculation of D. Moreover, it

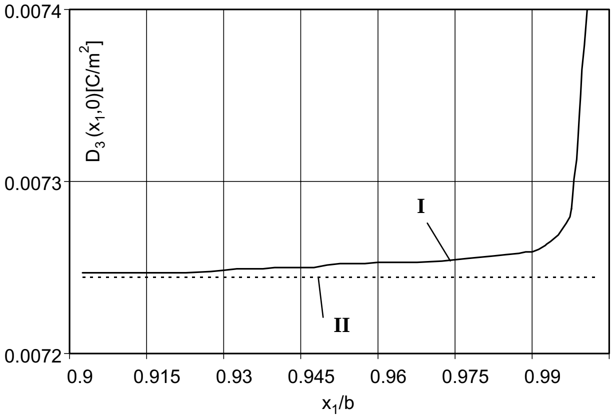

Fig. 2. Variation of $D_3^{(\mathrm{perm})}(x_1, 0)$ (line I) and D (line II) for the near crack tip region.

follows from the Table 1 that the IFs for the permeable crack (Herrmann and Loboda, 2000) and for the limited permeable crack with $\varepsilon_a = 4000\varepsilon_0$ almost coincide and this justifies again the above-mentioned assumption. It is worth to be mentioned that the values of D for the same bimaterial compound and for $\varepsilon_a = 10^{-6}\varepsilon_0$ (electrically impermeable crack), $\varepsilon_a = \varepsilon_0$ (air), $\varepsilon_a = 2.5\ \varepsilon_0$ (silicone oil) and $\varepsilon_a = 81\varepsilon_0$ (water) are equal 1.23×10^{-8} C/m^2, 0.00507 C/m^2, 0.00626 C/m^2 and 0.00721 C/m^2, respectively.

For the same bimaterial compound and for the above crack media, the values of the electric flux D through the crack and the IFs K_1, K_2 and K_4 are presented in Table 2 for the crack length of 2 mm. A strong mechanical loading ($\sigma = 10$ MPa, $\tau = 0$ MPa) and a weak external electric flux ($d = 0.001$ C/m^2) were chosen in this case. The electric permeability of the crack medium was represented in the form $\varepsilon_a = \varepsilon_r\varepsilon_0$. In Table 3, the corresponding values are given for the same materials, but for weak mechanical loading ($\sigma = 10^{-3}$ MPa, $\tau = 0$ MPa) and a strong electric flux ($d = 0.01$ C/m^2). It follows from these results that for the case of a weak electric flux (Table 2) the obtained IFs for all physical values of $\varepsilon_r \geqslant 1$ can be approximated by the corresponding results for the electrically permeable crack. On the other hand, for the case of a strong electric loading (Table 3) the IF K_2 and especially the electric intensity factor K_4 for air ($\varepsilon_r = 1$) and silicone oil ($\varepsilon_r = 2.5$) significantly differ from the corresponding values for the electrically permeable crack.

Table 1
Comparison of the IFs for the limited permeable crack ($\varepsilon_r = 4000$) with the exact values for permeable crack for bimaterial compound PZT4/PZT5 and $\sigma = 10$ MPa, $d = 0.001$ C/m^2

	K_1 (MPa$\sqrt{\text{m}}$)	K_2 (MPa$\sqrt{\text{m}}$)	K_4 (C/m$^{3/2}$)
Permeable crack	0.5597	0.03492	1.542×10^{-4}
Limited permeable crack with $\varepsilon_r = 4000$	0.5597	0.03494	1.546×10^{-4}

Table 2
The variation of the electrical flux and IFs for bimaterial compound PZT4/PZT5 under strong mechanical ($\sigma = 10$ MPa) and weak electrical ($d = 0.001$ C/m^2) loading

ε_r	$-D$ (C/m^2)	K_1 (MPa$\sqrt{\text{m}}$)	K_2 (MPa$\sqrt{\text{m}}$)	$K_4 \cdot 10^4$ (C/m$^{3/2}$)
10^{-6}	5.73×10^{-9}	0.5599	0.0273	0.561
1	0.00130	0.5597	0.0329	1.29
2.5	0.00154	0.5597	0.0340	1.42
81	0.00175	0.5597	0.0349	1.54
4000	0.00176	0.5597	0.0349	1.55

Table 3
The variation of the electrical flux and IFs for bimaterial compound PZT4/PZT5 under weak mechanical ($\sigma = 1000$ Pa) and strong electrical ($d = 0.01$ C/m^2) loading

ε_r	D (C/m^2)	K_1 (N/m$^{3/2}$)	K_2 (N/m$^{3/2}$)	$K_4 \cdot 10^8$ (C/m$^{3/2}$)
10^{-6}	3.21×10^{-8}	−956	4.36×10^4	5.61×10^4
1	0.00999	55.9	5.78	4.48
2.5	0.00999	56.0	4.21	2.47
81	0.00999	56.0	3.51	1.57
4000	0.00999	56.0	3.49	1.55

6. A crack in a homogeneous piezoelectric material

For the sake of clarity, consider now the special case of a homogeneous piezoelectric material. In this case, the solution obtained above can be evaluated for $\varepsilon_1 = 0$, $\gamma_1 = 1$. Moreover, since the relations $\cos(\alpha) = 1$, $\sin(\alpha) = 0$ are exact now, Eqs. (27) and (28) are exact as well and the assumption concerning D being constant along the interval $(-b, b)$ is valid without any error.

In case of a homogeneous material the formula (30) attains the form

$$\sigma_{33}^{(1)}(x_1,0) + m_{j4}D_3^{(1)}(x_1,0) + \mathrm{i}m_{j1}\sigma_{13}^{(1)}(x_1,0) = 2(\sigma_j^* - \mathrm{i}\tau_j^*)\frac{x_1}{\sqrt{x_1^2 - b^2}} + m_{j4}D \quad \text{for } x_1 \notin (-b,b). \tag{37}$$

Considering the real part of (37) for $j = 1$ and $j = 4$ leads to

$$\sigma_{33}^{(1)}(x_1,0) = \sigma\frac{x_1}{\sqrt{x_1^2 - b^2}}, \tag{38}$$

$$D_3^{(1)}(x_1,0) = (d - D)\frac{x_1}{\sqrt{x_1^2 - b^2}} + D \quad \text{for } x_1 \notin (-b,b). \tag{39}$$

Next, we introduce the stress and electric intensity factors as (Parton and Kudryavtsev, 1988)

$$K_1 = \lim_{x_1 \to b+0}\sqrt{2\pi(x_1 - b)}\sigma_{33}^{(1)}(x_1,0), \quad K_2 = \lim_{x_1 \to b+0}\sqrt{2\pi(x_1 - b)}\sigma_{13}^{(1)}(x_1,0), \tag{40}$$

$$K_4 = \lim_{x_1 \to b+0}\sqrt{2\pi(x_1 - b)}D_3^{(1)}(x_1,0). \tag{41}$$

The formulae (38) and (39) lead to the following expressions for these IFs:

$$K_1 = \sqrt{\pi b}\sigma, \quad K_2 = \sqrt{\pi b}\tau, \quad K_4 = \sqrt{\pi b}(d - D). \tag{42}$$

Note that definitions (40), (41) and the formulae (42) follow from the definitions (32), (33) and formulae (34), (35) provided $\varepsilon_1 = 0$, $\gamma_1 = 1$.

It is worth to mention that neither the stress $\sigma_{33}^{(1)}(x_1,0)$ nor the IF K_1 depend on the electric flux d. Moreover, the obtained expressions for the IFs completely coincide with the associated results of the paper by Hao and Shen (1994).

Using formulae (22) and (25) for a homogeneous material, the expressions for $[u_1(x_1,0)]$, $[u_3(x_1,0)]$ and $[\varphi(x_1,0)]$ take the form

$$\begin{aligned}
&[u_1(x_1,0)] = -m_{11}n_{11}^{-1}\tau\sqrt{b^2 - x_1^2},\\
&[u_3(x_1,0)] = [\vartheta_{11}\sigma + \vartheta_{12}(d - D)]\sqrt{b^2 - x_1^2},\\
&[\varphi(x_1,0)] = [\vartheta_{21}\sigma + \vartheta_{22}(d - D)]\sqrt{b^2 - x_1^2},
\end{aligned} \tag{43}$$

where $\vartheta_{11} = \Delta^{-1}(n_{44} - n_{14})$, $\vartheta_{12} = \Delta^{-1}(m_{14}n_{44} - m_{44}n_{14})$, $\vartheta_{21} = \Delta^{-1}(n_{13} - n_{43})$, $\vartheta_{22} = \Delta^{-1}(m_{44}n_{13} - m_{14}n_{43})$.

According to formulae (38), (39), the stress and the electric displacement can asymptotically approximated for $x_1 \to b + 0$ in the form

$$\sigma_{33}^{(1)}(x_1,0) = \sigma\sqrt{\frac{b}{2(x_1 - b)}}, \quad \sigma_{13}^{(1)}(x_1,0) = \tau\sqrt{\frac{b}{2(x_1 - b)}}, \quad D_3^{(1)}(x_1,0) = (d - D)\sqrt{\frac{b}{2(x_1 - b)}}. \tag{44}$$

The energy release rate (ERR) related to the point $x_1 = b$ can be introduced as (Parton and Kudryavtsev, 1988)

$$G = \lim_{\Delta l \to 0} \frac{1}{2\Delta l} \int_b^{b+\Delta l} \left\{ \sigma_{33}^{(1)}(x_1,0)[u_3(x_1-\Delta l,0)] + \sigma_{13}^{(1)}(x_1,0)[u_1(x_1-\Delta l,0)] + D_3^{(1)}(x_1,0)[\varphi(x_1-\Delta l,0)] \right\} \mathrm{d}x_1. \tag{45}$$

Taking into account that $D_3^{(1)}(x_1,0)$ is not singular at the point $x_1 \to b-0$ and using expressions (43), (44) leads to

$$G = \frac{\pi b}{4}[\vartheta_{11}\sigma^2 + (\vartheta_{12}+\vartheta_{21})\sigma(d-D) + \vartheta_{22}(d-D)^2 - m_{11}n_{11}^{-1}\tau^2]. \tag{46}$$

The numerical analysis of the obtained formulae has been performed for two types studies. First, $\sigma = 10$ MPa, $\tau = 0$, $d = 0.01$ C/m^2 and various values of ε_a were chosen. Second, $\sigma = 10$ MPa, $\tau = 0$, $\varepsilon_a = 8.85 \times 10^{-12}$ C/V m and several magnitudes of d have been applied. The obtained results are in very good agreement with the corresponding FEM values in the paper by Gruebner et al. (2003).

For a crack of 2-mm length in the homogeneous material PZT4 and for the same crack media as in the previous tables, the values of the electric flux D through the crack, the IF K_4 and the ERR G are presented in Table 4. A strong mechanical loading ($\sigma = 10$ MPa) and a weak external electric flux ($d = 0.001$ C/m^2) were chosen in this case and different electric permeabilities were defined by the coefficient ε_r. In Table 5, the corresponding results are given for the same material under a moderate mechanical loading ($\sigma = 1$ MPa) and a very strong external electric flux ($d = 0.03$ C/m^2). Similarly to the above conclusion concerning a bimaterial compound for all physical values of $\varepsilon_r \geqslant 1$, it can be seen that for a weak external electric flux (Table 4) the obtained electric IF K_4 and the ERR G are in good agreement with the corresponding results for the electrically permeable crack. However for the strong electric loading (Table 5), both the IF K_4 and the ERR G are in good agreement with the associated values for the electrically permeable crack only for water ($\varepsilon_r = 81$).

Finally, it is worth to discuss an interesting phenomenon connected with the nonlinearity of the problem studied. In Table 6, the results are presented for a mechanical loading and an electric flux increased proportionally 10 times with respect to the loading corresponding to Table 4. It seems that the values of IF

Table 4
The variation of the electrical flux, the electrical IF and the ERR for a crack in homogeneous material PZT4 under strong mechanical ($\sigma = 10$ MPa) and weak electrical ($d = 0.001$ C/m^2) loading

ε_r	$-D$ (C/m^2)	$K_4 \cdot 10^4$ (C/m$^{3/2}$)	G (N/m)
10^{-6}	6.01×10^{-13}	0.560	3.31
1	0.00119	1.23	3.61
2.5	0.00137	1.33	3.63
81	0.00152	1.42	3.63
4000	0.00153	1.42	3.63

Table 5
The variation of the electrical flux, the electrical IF and the ERR for a crack in homogeneous material PZT4 under moderate mechanical ($\sigma = 1$ MPa) and very strong electrical ($d = 0.03$ C/m^2) loading

ε_r	$-D$ (C/m^2)	$K_4 \cdot 10^4$ (C/m$^{3/2}$)	G (N/m)
10^{-6}	3.37×10^{-12}	16.8	-1.21×10^2
1	0.0265	1.97	-1.43
2.5	0.0292	0.435	-1.49×10^{-3}
81	0.0297	0.148	36.3×10^{-3}
4000	0.0297	0.142	36.3×10^{-3}

Table 6
Variations of the same values as in Table 4 for the electromechanical loading 10 times stronger ($\sigma = 100$ MPa, $d = 0.01$ C/m^2) than in this table

ε_r	$-D$ (C/m^2)	$K_4 \cdot 10^4$ (C/m$^{3/2}$)	G (N/m)
10^{-6}	6.01×10^{-13}	5.60	331
1	0.00418	7.95	345
2.5	0.00729	9.69	354
81	0.0148	13.8	363
4000	0.0153	14.2	363

K_4 and the ERR G increase proportionally only for the extreme cases of electrically permeable ($\varepsilon_r = 4000$) and electrically impermeable cracks ($\varepsilon_r = 10^{-6}$). On the other hand, these quantities together with D show nonlinear behavior for all cases of a limited permeable crack ($\varepsilon_r = 1.0, 2.5, 81$).

7. Conclusion

The problem of an interface crack in a piezoelectric bimaterial compound under electromechanical loading is considered. The crack is assumed to be completely opened and limited permeable electric conditions (11) at its faces are used. By assuming that the electric flux through the crack is constant, the problem is reduced to the Riemann problem (14), (15) which is solved exactly. The quadratic equation (28) with respect to the electric flux through the crack is formulated and the conclusion that only one root of this equation is physically meaningful is found. The analytical formulae (30) and (31) for stresses and electric flux as well as for the stress and electric intensity factors (34), (35) are derived. It is shown that the results for an electrically impermeable crack follow from the obtained formulae if the permeability ε_a of the crack medium tends to zero. Furthermore, the results found by means of those formulae for large values of ε_a are in very good agreement with the corresponding results for the electrically permeable crack (Herrmann and Loboda, 2000). In particular, the latter results confirm the validity of the assumption concerning the electric flux being constant along the crack faces.

As a special case of the obtained solution, a crack in a homogeneous piezoelectric material is analyzed. In this case, the electric flux is exactly constant along the crack faces and, therefore, the obtained results are exact as well. Formulae (38) and (39) for normal stresses and electric displacement become very simple in this case, and the stress and electric intensity factors (42) together with the energy release rate (46) are represented via the components of the electromechanical loading in analytical way.

Numerical results are given for the bimaterial compound PZT4/PZT5 and for crack in the homogeneous piezoelectric material PZT4. In the latter case, all computations demonstrate an excellent agreement with the corresponding results of the paper by Gruebner et al. (2003) performed by means of FEM. It follows from the obtained results that independent of the materials they are in good agreement with the corresponding results for the electrically permeable crack for weak electric fluxes. Increasing the electric flux (and the mechanical loading as well) leads to an increase of the difference between the mentioned values and the approximation of the crack medium as electrically permeable gets poor. Finally, it has been demonstrated by means of Tables 4 and 6 that the dependence of the electric flux through the crack and the dependence of the intensity factors and the ERR on the applied electromechanical loading are nonlinear.

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References

Dunn, M.L., 1994. The effect of crack faces boundary conditions on the fracture mechanics of piezoelectric solids. Eng. Fract. Mech. 48, 25–39.

Gao, C.F., Fan, W.X., 1999. Exact solution for the plane problem in piezoelectric materials with an elliptic hole or a crack. Int. J. Solids Struct. 36, 2527–2540.

Gao, C.F., Wang, M.Z., 2000. Collinear permeable cracks between dissimilar piezoelectric materials. Int. J. Solids Struct. 37, 4969–4986.

Govorukha, V.B., Loboda, V.V., 2000. Contact zone models for an interface crack in a piezoelectric material. Acta Mech. 140, 233–246.

Govorukha, V.B., Munz, D., Kamlah, M., 2000. On the singular integral equations approach to the interface crack problem for piezoelectric materials. Arch. Mech. 52, 247–273.

Gruebner, O., Kamlah, M., Munz, D., 2003. Finite element analysis of cracks in piezoelectric materials taking into account the permittivity of the crack medium. Eng. Fract. Mech. 70, 1399–1413.

Hao, T.H., Shen, Z.Y., 1994. A new electric boundary condition of electric fracture mechanics and its applications. Eng. Fract. Mech. 47, 793–802.

Herrmann, K.P., Loboda, V.V., 2000. Fracture mechanical assessment of electrically permeable interface cracks in piezoelectric bimaterials by consideration of different contact zone models. Arch. Appl. Mech. 70, 127–143.

Herrmann, K.P., Loboda, V.V., Govorukha, V.B., 2001. On contact zone models for an interface crack with electrically insulated crack surfaces in a piezoelectric bimaterial. Int. J. Fract. 111, 203–227.

Kogan, L., Hui, C.Y., Molkov, V., 1996. Stress and induction field of a spheroid inclusion or a penny shaped crack in a transversely isotropic piezoelectric material. Int. J. Solids Struct. 33, 2719–2737.

Muskhelisvili, N.I., 1953. Singular Integral Equations. Noordhoff, Groningen.

Pak, Y.E., 1992. Linear electro-elastic fracture mechanics of piezoelectric materials. Int. J. Fract. 54, 79–100.

Parton, V.Z., 1976. Fracture mechanics of piezoelectric materials. Acta Astronaut. 3, 671–683.

Parton, V.Z., Kudryavtsev, B.A., 1988. Electromagnetoelasticity. Gordon and Breach, New York.

Sosa, H.A., Khutoryansky, N., 1996. New developments concerning piezoelectric materials with defects. Int. J. Solids Struct. 33, 3399–3414.

Suo, Z., Kuo, C.M., Barnet, D.M., Willis, J.R., 1992. Fracture mechanics for piezoelectric ceramics. J. Mech. Phys. Solids 40, 739–765.

Wang, B.L., May, Y.-W., 2003. On the electric boundary conditions on the crack surfaces in piezoelectric ceramics. Int. J. Engng Sci. 41, 633–652.

Zhang, T.Y., Qian, C.F., Tong, P., 1998. Linear electro-elastic analysis of a cavity or a crack in a piezoelectric material. Int. J. Solids Struct. 35, 2121–2149.